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# Engineering Notes

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## Rotations as Double Reflections and Geometrical Derivation of Euler–Rodrigues Parameters

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### I. Introduction

ALTHOUGH the Euler–Rodrigues parameters [1] and the associated Hamiltonian quaternion algebra [2] have been available for more than a century and a half, their use to represent rotations in engineering problems is not as overwhelming as their mathematical elegance and numerical superiority [3,4] would justify. Where matrix and vector algebra have found broad adoption in engineering science, replacing elaborate parametric descriptions of three-dimensional kinematic and dynamic equations that were used well into the 19th century, the spatial orientation of bodies is most often expressed in terms of either the three Euler angles or a three-by-three rotation matrix. The nonlinearities and possible singularity of Euler angles prevent the construction of a satisfactory algebra of rotations; rotation matrices do not have the favorable interpretation of the Euler angles and introduce a large computational overhead as only three of the nine parameters are independent. One of the inevitable causes for the quaternion's unpopularity is the incompatibility of its parameters with the way orientation is perceived in daily life. Where position and translation are observed by the same three parameters that form the components of the corresponding vector in three-dimensional space, orientation is primarily perceived as a set of resulting angles and not as a single rotation from some reference orientation.

Without ignoring the importance of this drawback, quaternions have suffered from some alleged and unnecessary disadvantages as well, including some common misconceptions in their interpretation [5]. For the most part, these are related to the double angles of a quaternion rotation with respect to the angle in the quaternion representation [3], and the resulting $4\pi$ periodicity of the quaternion operator. Furthermore, literature on the use of quaternions for representing attitude is highly scattered, as noted in the review paper by Phillips et al.[3]. Finally, in almost all of the literature that is known to the author on the use of quaternions in kinematics, quaternions are introduced by their algebra and the definition of the Euler–Rodrigues parameters first, and are subsequently shown (if proof is given at all) to perform the rotation of a vector. Although this procedure is mathematically impeccable in order to serve as a *proof*, it is only somewhat appropriate as a *derivation* that should not only prove the correctness of the formulae but also lead to full comprehension of the concept.

This paper presents an alternative introduction to the Euler–Rodrigues parameters and their multiplication rules. The derivation is fully geometrical; at no point is Hamiltonian quaternion algebra used. The original derivation of the four-component parameterization of rotations that allows the determination of a single rotation from the product of two others (the Euler construction) by Rodrigues [1] was published in 1840, four years before the invention of quaternions by Hamilton [2]. Obviously, Rodrigues' construction does not depend on quaternion algebra. The difference between the derivation that is presented here and Rodrigues' Euler construction lies in the mathematical tools that are used. Instead of applying spherical trigonometry to the poles of the rotations [1,6,7], the derivation that is presented here uses only plain vector algebra. It is based on the recognition that a rotation is equal to a pair of reflections by intersecting mirrors, which is a well-known result in Clifford algebra as it is used for physical science [8,9]. After the geometry of rotations has been discussed to formulate the goal of the following considerations, the effect of a double reflection is worked out in detail. It is shown that many characteristics of the quaternion that are often perceived as peculiar, for example the $4\pi$ periodicity that was mentioned earlier, can already be observed in the construction of two reflections. Using the insights into double reflections, an intuitive parameterization of a rotation is derived, which coincides with the Euler–Rodrigues parameters. The final step is the derivation of the expressions that transform the parameters for two successive rotations into those of a single rotation, constituting a proof for the Euler construction.

### II. Geometry of Rotations

The generic objective of a rotation operator is the *conical transformation* of a vector in three-dimensional space, also referred to as the conical rotation. Notwithstanding its importance as the key operator for which direction cosine matrices, Euler angles, and quaternions are mere representations, the conical transformation is only seldom discussed in literature. A comprehensive treatment is presented by Altmann [6], to which an alternative construction is presented here.

Figure 1 shows the rotation operation on an arbitrary vector $\mathbf{v}$. The vector is rotated over an angle $\alpha$ about the axis of rotation, which is represented by the unit vector $\mathbf{a}$, resulting in the vector $\mathbf{v}''$. An expression for $\mathbf{v}''$ in terms of $\mathbf{v}$, $\mathbf{a}$, and $\alpha$ is obtained by separating $\mathbf{v}$ into its components parallel and perpendicular to $\mathbf{a}$. Because $\mathbf{a}$ has unit length, the component along the axis of rotation $\mathbf{v}_\parallel$ equals $(\mathbf{v}\cdot\mathbf{a})\mathbf{a}$; the perpendicular component is the difference of $\mathbf{v}$ and the parallel component $\mathbf{v}_\parallel$. Because the parallel component is unaffected by the rotation, the conical transformation can be split into a vector sum of $\mathbf{v}_\parallel$ and the result of a *rectangular rotation* of $\mathbf{v}_\perp$ in the plane perpendicular to the axis of rotation.

The plane perpendicular to $\mathbf{a}$ is spanned by the two orthogonal vectors $\mathbf{v}_\perp = \mathbf{v} - (\mathbf{v}\cdot\mathbf{a})\mathbf{a}$ and $\mathbf{a}\times\mathbf{v}$, both of which have length $|\mathbf{v}|\sin\gamma$, where $\gamma$ is the angle between $\mathbf{v}$ and $\mathbf{a}$. Because $\mathbf{a}\times\mathbf{v}$ points in the direction of a positive rotation over 90 deg about $\mathbf{a}$, the

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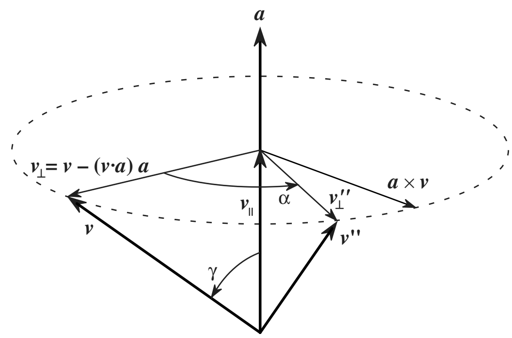


Fig. 1 **The conical transformation.**

component of the rotated vector $\mathbf{v}''$ in the plane is given by

$$\mathbf{v}''_{\perp} = (\mathbf{v} - (\mathbf{v}\cdot\mathbf{a})\mathbf{a})\cos\alpha + (\mathbf{a}\times\mathbf{v})\sin\alpha$$

The complete rotated vector is the sum of the rotated perpendicular and the invariant parallel components:

$$\mathbf{v}'' = \mathbf{v}_{\parallel} + \mathbf{v}''_{\perp} = (\mathbf{v}\cdot\mathbf{a})\mathbf{a} + (\mathbf{v} - (\mathbf{v}\cdot\mathbf{a})\mathbf{a})\cos\alpha + (\mathbf{a}\times\mathbf{v})\sin\alpha$$
$$= \mathbf{v}\cos\alpha + (\mathbf{v}\cdot\mathbf{a})\mathbf{a}(1-\cos\alpha) + (\mathbf{a}\times\mathbf{v})\sin\alpha \tag{1}$$

Equation (1) is the geometrical representation of a generic rotation. It is to be shown that any algebraic operator for rotations satisfies Eq. (1).

## III. Double Reflections

The double reflection of an arbitrary vector by two intersecting mirrors is shown in Fig. 2. The mirrors $S_1$ and $S_2$ are defined by their unit normal vectors $\sigma_1$ and $\sigma_2$, respectively. The product of the reflection of the vector $\mathbf{v}$ in the first mirror $S_1$ is denoted as $\mathbf{v}'$. The vector $\mathbf{v}''$ is the reflection of $\mathbf{v}'$ in the second mirror $S_2$. The figure leads one to believe that such a double reflection equals a conical transformation about the intersection of the two mirrors, which will be shown in this section by demonstrating its equivalence to Eq. (1).

### A. Rotation Angle

Similar to the case of the conical transformation, the component of $\mathbf{v}$ that is parallel to the intersection of the two mirrors (hence the component that lies both in $S_1$ and $S_2$) the double reflection only affects the part of $\mathbf{v}$ that lies in the plane perpendicular to the intersection. A top view of this plane is shown in Fig. 3. As a reflection does not change the length of a vector, the effect of the two reflections on the vector $\mathbf{w}$ can easily be expressed in polar coordinates. The argument of the first reflection $\mathbf{w}'$ equals twice the angle between the original vector and the first mirror $S_1$, indicated by $\psi_1$. If $\phi$ denotes the angle between the two mirrors, the angle $\psi_2$ between the first reflection $\mathbf{w}'$ and the second mirror $S_2$ equals $\phi - \psi_1$, in which all angles must adhere to the sign convention of

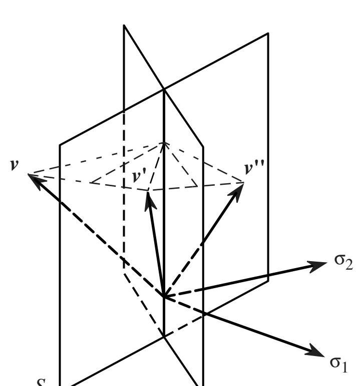


Fig. 2 **Rotation as the product of two reflections.**

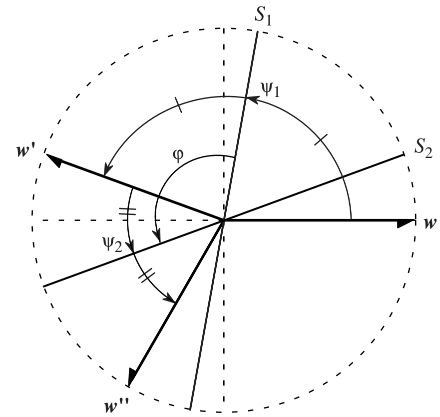


Fig. 3 **Angles for a double reflection.**

being positive in the direction of the arrow. Because the argument of the second reflection $\mathbf{w}''$ equals $2\psi_1 + 2\psi_2$, the angle between the original vector $\mathbf{w}$ and the rotated vector $\mathbf{w}''$ simply equals twice the angle between the mirrors:

$$\alpha = 2\phi \tag{2}$$

The first important conclusion from this result is the rotation angle's independence from the angle $\psi_1$ between the vector $\mathbf{w}$ and the pair of mirrors. Only the dihedral angle $\phi$ between the two mirrors determines the rotation angle; it is unaffected by the actual orientation of the two mirrors with respect to the vector. This result can be extended back to the spatial case of Fig. 2 as long as the axis of rotation is unchanged, which means that the plane in which the rotation is rectangular is unchanged. Any pair of mirrors with the same intersection and the same angle between the two mirror surfaces will, therefore, produce the same rotation. The pair of mirrors itself can freely be rotated about the axis of rotation without influencing the effect of the operation.

A second conclusion that can be drawn from the previous analysis is that the two reflections do not commute. If $\mathbf{w}$ is reflected in $S_2$ first, the argument of its image $\mathbf{w}'_{S_2,S_1}$ becomes $2(\psi_1 + \phi)$; the angular distance to the first mirror is now $-2\phi - \psi_1$, which results in an image $\mathbf{w}''_{S_2,S_1}$ with the argument $-2\phi$. A reflection in $S_2$ followed by a reflection in $S_1$, therefore, results in a rotation over the same angle but in the other direction than a reflection in $S_1$ followed by a reflection in $S_2$.

As a third result, Fig. 3 can be used to analyze the range of the rotation operator. If the angle between the original vector and a mirror surface is defined as the smallest angle between the vector and the mirror, each of the angles $\psi_1$ and $\psi_2$ is limited by

$$-\frac{\pi}{2} < \psi \le \frac{\pi}{2} \tag{3}$$

in which the sign of the angle is in line with the positive direction in Fig. 3, and the interval is left open at the side of $-\pi/2$ to avoid ambiguity in case of a vector that is perpendicular to the mirror. As $\phi = \psi_1 + \psi_2$, the range of $\phi$ is given by

$$-\pi < \phi \le \pi \tag{4}$$

and, consequently, for the angle of rotation $\alpha = 2\phi$

$$-2\pi < \alpha \le 2\pi \tag{5}$$

This is an illustration of the $4\pi$ periodicity of the rotation operator. Although the angle between the vector and each mirror was defined without ambiguity, the rotation operation that is the product of two reflections covers a $4\pi$ range. The reason for this is that the definition of $\psi$ according to Eq. (3) prescribes the direction in which the angle between the vector and the mirror is measured. Although it restricts the absolute angle between each vector and the corresponding mirror to the smallest value possible, the resulting angle between the two

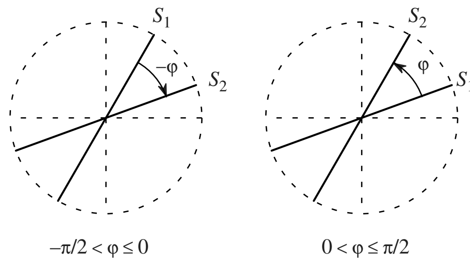


**Fig. 4 Angles for nondirectional mirrors.**

mirrors can become larger than $\pi/2$, as can easily be verified in Fig. 3. The mirrors have effectively become directional. By prescribing the direction in which the path from a vector to its image along the circle in Fig. 3 passes the mirror, a difference has been introduced between a mirror at angle $\psi$ and a mirror at angle $\psi + \pi$, although both physically perform the same reflection. This difference leads to a difference between a vector's image at angle $\alpha$ and at $\alpha + 2\pi$.[†]

The $4\pi$ periodicity of the rotation operator can be resolved by eliminating the directionality of mirrors. The first from the aforementioned conclusions states that only the dihedral angle of the mirrors and not their orientation with respect to the vector is important. Therefore, the angles $\psi$ between the vector and the mirrors need not be specified at all, and the mirrors' dihedral angle can be restricted to a physically meaningful range of $-\pi/2 < \phi \leq \pi/2$ instead of the range given by Eq. (4). A positive angle represents the case in which the smallest angle to go from $S_1$ to $S_2$ rotates in the counterclockwise direction; a negative angle represents the case in which the smallest angle from $S_1$ to $S_2$ rotates clockwise. Both cases are depicted in Fig. 4. The figure also illustrates how the sign of the rotation angle changes when the two mirrors are interchanged, as was described earlier.

Although nondirectional mirrors would be entirely sufficient for modeling rotations in physical three-dimensional space, geometrical analyses are made with directional mirrors. The reason for this is the argument presented earlier, which is based on Eq. (3). Rather than defining the angle between a pair of mirrors as in Fig. 3, each mirror is separately defined by a unique orientation. In practice, this orientation is formulated in terms of the mirror's unit normal vector $\sigma$. Although $\sigma$ and $-\sigma$ represent the same physical mirror, the two vectors are obviously not equal. Mirrors that are defined by their normal vectors are, therefore, inherently directional, and the resulting rotation operator has a periodicity of $4\pi$.

### B. Rotation Parameterization

Returning to Fig. 2, an expression for the image $\mathbf{v}''$ is sought in terms of the vector $\mathbf{v}$ and the two mirrors, identified by their respective normals $\sigma_1$ and $\sigma_2$. However, the two individual normals should not appear in the equation because the position of the separate mirrors is of no importance to the rotation. Instead, the distinguishing parameters are the angle between the mirrors and their line of intersection. Because the latter is by definition the line that lies in both mirror planes, it must be perpendicular to both of the mirror normals. The rotation axis is therefore easily identified by the vector product of the two normals $\sigma_1 \times \sigma_2$. Comparing Fig. 2 with Fig. 4 shows that the positive direction of the vector product corresponds to a right-hand rotation over a positive angle. Because the two normals are of unit length, the length of the vector product equals the sine of the angle between the two mirrors:

$$|\sigma_1 \times \sigma_2| = \sin\phi$$

For nondirectional mirrors, where $-\pi/2 < \phi \leq \pi/2$, the sine of each possible $\phi$ is unique, and the vector product completely determines the rotation. For directional mirrors, however, the sine does not uniquely identify the mirrors' dihedral angle and, therefore, does not uniquely determine the rotation angle. The latter can be fixed when the cosine of the angle between the two mirrors is also known, which follows directly from the scalar product of the two mirror normals:

$$\sigma_1 \cdot \sigma_2 = \cos\phi \tag{6}$$

A sufficient parameterization of a rotation by a double reflection is, therefore, given by the combination of the vector and the scalar product of the unit normal vectors for the two mirrors.

As can be verified from Fig. 5, the image of a reflection satisfies

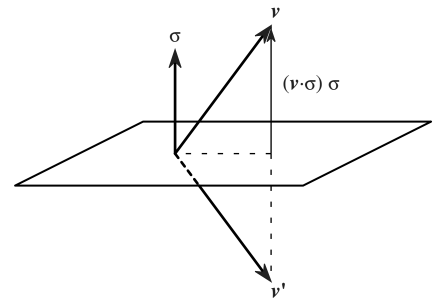


**Fig. 5 Simple reflection.**

$$\mathbf{v}' = \mathbf{v} - 2(\mathbf{v}\cdot\sigma)\sigma$$

Therefore, for a double reflection in $\sigma_1$ and $\sigma_2$

$$\begin{aligned}\mathbf{v}'' &= \mathbf{v}' - 2(\mathbf{v}'\cdot\sigma_2)\sigma_2 \\ &= \mathbf{v} - 2(\mathbf{v}\cdot\sigma_1)\sigma_1 - 2\{[\mathbf{v} - 2(\mathbf{v}\cdot\sigma_1)\sigma_1]\cdot\sigma_2\}\sigma_2 \\ &= \mathbf{v} - 2(\mathbf{v}\cdot\sigma_1)\sigma_1 - 2(\mathbf{v}\cdot\sigma_2)\sigma_2 + 4(\mathbf{v}\cdot\sigma_1)(\sigma_1\cdot\sigma_2)\sigma_2\end{aligned} \tag{7}$$

Equation (7) is now reformulated in order to replace any individual occurrence of the mirror vectors $\sigma$ by an expression in which only the scalar or vector product of both appears. This can be achieved through repeated application of Lagrange's formula $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. First, it is used to introduce the following substitutions:

$$\sigma_1 = (\sigma_1\cdot\sigma_2)\sigma_2 - (\sigma_1\times\sigma_2)\times\sigma_2$$

$$\sigma_2 = (\sigma_1\cdot\sigma_2)\sigma_1 + (\sigma_1\times\sigma_2)\times\sigma_1$$

which turn Eq. (7) into

$$\begin{aligned}\mathbf{v}'' = \mathbf{v} &- 2(\mathbf{v}\cdot\sigma_1)[(\sigma_1\cdot\sigma_2)\sigma_2 - (\sigma_1\times\sigma_2)\times\sigma_2] \\ &+ -2(\mathbf{v}\cdot\sigma_2)[(\sigma_1\cdot\sigma_2)\sigma_1 + (\sigma_1\times\sigma_2)\times\sigma_1] \\ &+ 4(\mathbf{v}\cdot\sigma_1)(\sigma_1\cdot\sigma_2)\sigma_2\end{aligned}$$

By grouping the terms with the vector product $\sigma_1 \times \sigma_2$ and the scalar product $\sigma_1 \cdot \sigma_2$, this is reorganized into:

$$\begin{aligned}\mathbf{v}'' &= \mathbf{v} + 2(\sigma_1\times\sigma_2)\times[(\mathbf{v}\cdot\sigma_1)\sigma_2 - (\mathbf{v}\cdot\sigma_2)\sigma_1] \\ &\quad + 2(\sigma_1\cdot\sigma_2)[(\mathbf{v}\cdot\sigma_1)\sigma_2 - (\mathbf{v}\cdot\sigma_2)\sigma_1]\end{aligned} \tag{8}$$

Lagrange's formula is used again to replace the bracketed parts of Eq. (8) by $(\sigma_1\times\sigma_2)\times\mathbf{v}$:

$$\mathbf{v}'' = \mathbf{v} + 2(\sigma_1\times\sigma_2)\times[(\sigma_1\times\sigma_2)\times\mathbf{v}] + 2(\sigma_1\cdot\sigma_2)[(\sigma_1\times\sigma_2)\times\mathbf{v}] \tag{9}$$

which is the desired form of Eq. (7), where the two mirror normals only appear in combination of both as a scalar or vector product.

It can now be verified that the parameterization of a rotation in terms of the scalar and vector products of the unit normal vectors of two mirrors is indeed equivalent to the parameterization by a rotation axis and an angle as shown in Eq. (1). Let $\mathbf{a}$ be the unit vector in the positive direction of $\sigma_1 \times \sigma_2$, so that

[†]The $4\pi$ periodicity of the rotation operator, considered a peculiarity in engineering, has a direct application in physics. Spinors invert under a rotation over $2\pi$, causing only a $4\pi$ rotation to represent the identity [6,8,9].

$$\sigma_1 \times \sigma_2 = \sin\phi \cdot \mathbf{a} \tag{10}$$

Substituting this equation and Eq. (6) into Eq. (9) yields

$$\mathbf{v}'' = \mathbf{v} + 2\sin\phi \cdot \mathbf{a} \times (\sin\phi \cdot \mathbf{a} \times \mathbf{v}) + 2\cos\phi(\sin\phi \cdot \mathbf{a} \times \mathbf{v}) \tag{11}$$

Using Lagrange's formula and introducing double angles, Eq. (11) provides

$$\mathbf{v}'' = \mathbf{v} + (1 - \cos 2\phi)[(\mathbf{a} \cdot \mathbf{v})\mathbf{a} - \mathbf{v}] + \sin 2\phi \cdot \mathbf{a} \times \mathbf{v}$$

which is equal to Eq. (1) if the rotation angle $\alpha$ is taken as $2\phi$, which was found earlier in Eq. (2). This completes the proof that a double reflection equals a conical transformation, with the intersection of the two mirrors $\sigma_1 \times \sigma_2$ as the axis of rotation and the dihedral angle between the two mirror normals as the angle of rotation.

## IV. Joint Rotations

Now that the parameterization of a rotation in terms of the vector and scalar products of the unit normal vectors of two intersecting mirrors has been established, it is possible to analyze the product of two rotations. First, it is shown that the product of any two rotations can be represented by a single rotation; then, the multiplication rules for the rotation operator are derived.

### A. The Euler Construction

The original proof that the generic motion of an object with one point fixed is a rotation and that, therefore, any two successive rotations are equivalent to a single rotation, is due to Euler [10]. The interpretation using double reflections can be used to prove Euler's theorem without reproducing the full Euler construction, as is done in this subsection. The reasoning itself is the same as is given by Altmann [6].

Because a rotation is the product of two reflections in intersecting mirrors, the product of two rotations can be replaced by a succession of four reflections. In the tetrahedron of Fig. 6, $OA$ represents the axis of the first rotation. Any two surfaces through $OA$ at a dihedral angle of half the desired rotation angle (positive in the sense of $O \to A$) satisfy the requirements for producing a rotation about $OA$. $OB$ is the axis of the second rotation, and, similar to the first rotation, any two surfaces through $OB$ can be used for the second pair of reflections. Now the second mirror for the first rotation and the first mirror for the second rotation are both chosen to be the plane $OAB$, thus fixing the first mirror through $OA$ and the fourth mirror through $OB$ by the respective required dihedral angles. Because both go through $O$, the first and the fourth mirror intersect as well. This intersection is indicated as $OC$. The two rotations are, therefore, equal to a reflection in $OAC$, followed by two reflections in $OAB$, and finally a reflection in $OBC$. Because the two middle reflections cancel, the product of two rotations is a reflection in $OAC$ followed by a reflection in $OBC$, which equals a single rotation about $OC$, thus proving Euler's theorem.

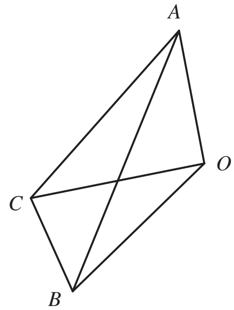


**Fig. 6 Mirror interpretation of the Euler construction.**

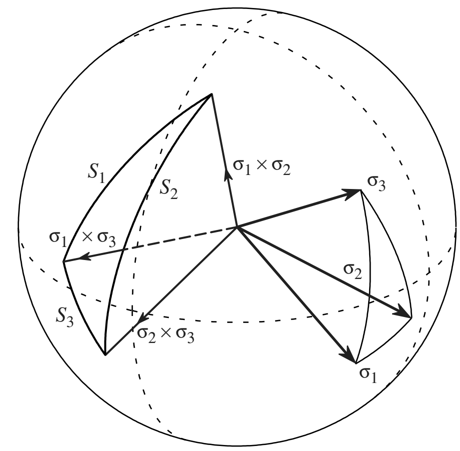


**Fig. 7 Joint rotations.**

### B. Operator product

Figure 7 shows the three mirror surfaces of the tetrahedron again, now inside a unit sphere. $S_1$ is the mirror $OAC$ of the first reflection; it is identified by its normal vector $\sigma_1$. $S_2$ is the common mirror $OAB$ for both reflections and is identified by $\sigma_2$. $S_3$ is the second mirror $OBC$ of the second rotation and is identified by $\sigma_3$. The first axis of rotation $\sigma_1 \times \sigma_2$ points along the intersection of $S_1$ and $S_2$, which is the segment $OA$ in Fig. 6. Similarly, the second axis of rotation $\sigma_2 \times \sigma_3$ points along the intersection of $S_2$ and $S_3$, which is the segment $OB$ in Fig. 6. As discussed in the previous subsection, the product of the two operations is a rotation about the intersection $OC$ of $S_1$ and $S_3$, hence about the axis $\sigma_1 \times \sigma_3$. To find a closed-form solution for the joint rotation, the question is, therefore, to find the vector and scalar products of $\sigma_1$ and $\sigma_3$ in terms of the vector and scalar products of $\sigma_1$ and $\sigma_2$ and of $\sigma_2$ and $\sigma_3$, respectively.

Again, the solution relies heavily on Lagrange's formula. Because $\sigma_2 = (\sigma_1 \times \sigma_2) \times \sigma_1 + (\sigma_1 \cdot \sigma_2)\sigma_1$ and $\sigma_3 = (\sigma_2 \times \sigma_3) \times \sigma_2 + (\sigma_2 \cdot \sigma_3)\sigma_2$, then

$$\begin{aligned}\sigma_1 \cdot \sigma_3 &= \sigma_1 \cdot \{(\sigma_2 \times \sigma_3) \times \sigma_2 + (\sigma_2 \cdot \sigma_3)\sigma_2\} \\ &= \sigma_1 \cdot \{(\sigma_2 \times \sigma_3) \times [(\sigma_1 \times \sigma_2) \times \sigma_1 + (\sigma_1 \cdot \sigma_2)\sigma_1] \\ &\quad + (\sigma_2 \cdot \sigma_3)\sigma_2\} = \sigma_1 \cdot \{(\sigma_2 \times \sigma_3) \times [(\sigma_1 \times \sigma_2) \times \sigma_1]\} \\ &\quad + \sigma_1 \cdot \{(\sigma_2 \times \sigma_3) \times (\sigma_1 \cdot \sigma_2)\sigma_1\} + (\sigma_1 \cdot \sigma_2)(\sigma_2 \cdot \sigma_3)\end{aligned} \tag{12}$$

Because the middle term is the scalar product of $\sigma_1$ and a vector product of $\sigma_1$ itself, which is zero as a vector product is perpendicular to its two components, Eq. (12) can be simplified to

$$\sigma_1 \cdot \sigma_3 = \sigma_1 \cdot \{(\sigma_2 \times \sigma_3) \times [(\sigma_1 \times \sigma_2) \times \sigma_1]\} + (\sigma_1 \cdot \sigma_2)(\sigma_2 \cdot \sigma_3) \tag{13}$$

Lagrange's formula is applied to the embraced triple product of $(\sigma_2 \times \sigma_3)$, $(\sigma_1 \times \sigma_2)$, and $\sigma_1$ in Eq. (13):

$$\begin{aligned}\sigma_1 \cdot \sigma_3 &= \sigma_1 \cdot \{[(\sigma_2 \times \sigma_3) \cdot \sigma_1](\sigma_1 \times \sigma_2) \\ &\quad - [(\sigma_1 \times \sigma_2) \cdot (\sigma_2 \times \sigma_3)]\sigma_1\} + (\sigma_1 \cdot \sigma_2)(\sigma_2 \cdot \sigma_3)\end{aligned} \tag{14}$$

Because the vector $\sigma_1 \times \sigma_2$ is perpendicular to $\sigma_1$, their scalar product is zero, and Eq. (14) reduces to

$$\sigma_1 \cdot \sigma_3 = -(\sigma_1 \times \sigma_2) \cdot (\sigma_2 \times \sigma_3) + (\sigma_1 \cdot \sigma_2)(\sigma_2 \cdot \sigma_3) \tag{15}$$

The scalar product for the joint rotation thus equals the product of the scalar products for each rotation, minus the scalar product of the two vector products for each rotation.

The vector product for the joint rotation is also obtained by substitution of $\sigma_3$ with Lagrange's formula:

$$\sigma_1 \times \sigma_3 = \sigma_1 \times [(\sigma_2 \times \sigma_3) \times \sigma_2 + (\sigma_2 \cdot \sigma_3)\sigma_2]$$
$$= \sigma_1 \times [(\sigma_2 \times \sigma_3) \times \sigma_2] + (\sigma_2 \cdot \sigma_3)(\sigma_1 \times \sigma_2)$$
$$= (\sigma_1 \cdot \sigma_2)(\sigma_2 \times \sigma_3) - [\sigma_1 \cdot (\sigma_2 \times \sigma_3)]\sigma_2 + (\sigma_2 \cdot \sigma_3)(\sigma_1 \times \sigma_2) \tag{16}$$

With $[\sigma_1 \cdot (\sigma_2 \times \sigma_3)]\sigma_2 = (\sigma_1 \times \sigma_2) \times (\sigma_2 \times \sigma_3) + [\sigma_2 \cdot (\sigma_2 \times \sigma_3)]\sigma_1$ and $\sigma_2 \cdot (\sigma_2 \times \sigma_3) = 0$, this becomes

$$\sigma_1 \times \sigma_3 = (\sigma_2 \cdot \sigma_3)(\sigma_1 \times \sigma_2) + (\sigma_1 \cdot \sigma_2)(\sigma_2 \times \sigma_3) + (\sigma_2 \times \sigma_3) \times (\sigma_1 \times \sigma_2) \tag{17}$$

The joint rotation's vector product is thus composed from the multiplication of the scalar product for one rotation, the vector product for the other rotation, and the vector product of both rotations' vector product.

## V. Conclusions

With Eq. (15) and (17), a parameterization of the rotation operator in terms of the scalar and vector products of two unit vectors is extended with a closed-form set of multiplication rules. These rules can be used to generate the parameters for the single rotation that is the product of any two rotations in terms of the same parameters. The rules follow directly from the interpretation of a rotation as the result of two reflections, where the unit vectors used in the parameterization are the normal vectors to the two mirrors. The interpretation also explains why the rotation angle is twice the angle that appears in the equations and why the operator has a periodicity of $4\pi$.

Using the Euler construction and spherical trigonometry, Rodrigues [1] introduced a four-term parameterization for rotations and solved the remaining problem of finding the parameters of a joint rotation. If $\alpha$ is the angle of rotation and $a_x$, $a_y$, and $a_z$ are the three components of a unit vector in the positive direction of rotation, these parameters are

$$\cos\frac{\alpha}{2}, \qquad \sin\frac{\alpha}{2}a_x, \qquad \sin\frac{\alpha}{2}a_y, \qquad \sin\frac{\alpha}{2}a_z \tag{18}$$

which are now known as the Euler–Rodrigues parameters or Euler symmetric parameters [11]. Comparison of Eq. (18) with Eq. (2) for the rotation angle, Eq. (6) for the scalar product, and Eq. (10) for the vector product shows that these parameters are equivalent to the mirror-based parameterization through $\sigma_1 \cdot \sigma_2$ and $\sigma_1 \times \sigma_2$, which is discussed in this paper.

Using the notation of Altmann [6] here, Rodrigues gave the following solution[‡] for the product $\mathcal{R}(\gamma\mathbf{c})$ of a rotation $\mathcal{R}(\beta\mathbf{b})$ followed by a rotation $\mathcal{R}(\alpha\mathbf{a})$:

$$\cos\frac{\gamma}{2} = \cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}(\mathbf{a}\cdot\mathbf{b})$$
$$\sin\frac{\gamma}{2}\cdot\mathbf{c} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cdot\mathbf{a} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cdot\mathbf{b} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}(\mathbf{a}\times\mathbf{b})$$

With the first rotation $\mathcal{R}(\beta\mathbf{b})$ being equal to $\mathcal{R}(2(\sigma_1 \cdot \sigma_2), (\sigma_1 \times \sigma_2)/|\sigma_1 \times \sigma_2|)$, the second rotation $\mathcal{R}(\alpha\mathbf{a})$ being equal to $\mathcal{R}(2(\sigma_2 \cdot \sigma_3), (\sigma_2 \times \sigma_3)/|\sigma_2 \times \sigma_3|)$, and the joint rotation $\mathcal{R}(\gamma\mathbf{c})$ being equal to $\mathcal{R}(2(\sigma_1 \cdot \sigma_3), (\sigma_1 \times \sigma_3)/|\sigma_1 \times \sigma_3|)$, Rodrigues' multiplication rules are the same as those presented in the previous section.

Finally, consider the algebraic interpretation of quaternions as the combination of a scalar and a pseudovector $\mathbb{A} = [a, \mathbf{a}]$, and the product of two quaternions in this interpretation [6]:

$$[a, \mathbf{a}][b, \mathbf{b}] = [ab - \mathbf{a}\cdot\mathbf{b}, b\mathbf{a} + a\mathbf{b} + \mathbf{a}\times\mathbf{b}] \tag{19}$$

When the first quaternion $\mathbb{B}$ is chosen as $[\sigma_2 \cdot \sigma_3, \sigma_2 \times \sigma_3]$ and the second quaternion $\mathbb{A}$ is chosen as $[\sigma_1 \cdot \sigma_2, \sigma_1 \times \sigma_2]$, the algebraic multiplication of Eq. (19) coincides with the rules from Eqs. (15) and (17). This shows that for two quaternions $\mathbb{A}$ and $\mathbb{B}$, the quaternion product $\mathbb{AB}$ represents the joint operation of the rotation identified by $\mathbb{B}$, followed by the rotation that is identified by $\mathbb{A}$.[§] More important, the analysis of the previous sections has shown that rotations are uniquely identified by the combination of a scalar and a pseudovector (being the scalar and the vector product of the normals of a pair of mirrors) and that joint rotations can easily be computed by a set of multiplication rules for the scalar/vector combination. Equation (19) is a practical convention to write down these rules, rather than an artificial mathematical construction that happens to be applicable to rotations.

It is not the intention of this paper to introduce a new parameterization for rotations, as the scalar/vector product formulation used here is effectively equal to the well-known Euler–Rodrigues parameters. For numerical computations as used in simulation or attitude determination and control, the purely algebraic approach of Hamiltonian quaternions is most suitable; whenever their components must be interpreted as finite rotations, the interpretation with Euler–Rodrigues parameters can be used. The intention of this paper is to show that the applicability of quaternion algebra to rotations is not a mere accident, but that the multiplication rules for quaternions can be derived from the geometry of rotations without using spherical trigonometry or quaternion algebra itself. Being able to think of the Euler–Rodrigues parameters as the scalar product and the vector product of two mirrors helps one to understand the reason for working with half angles and why two different quaternions produce the same physical rotation. It is hoped that this contributes to an increased appreciation for the elegance of quaternions and to a larger acceptance of quaternion algebra in the aerospace community.

## References

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[‡]With the exception of the vector notation, which did not exist at the time [6].

[§]The sequence of rotations, or the reverse order of multiplication of quaternions, as shown here follows the original conventions of Hamilton. An alternative representation is described by Shuster [4]; the choice between the two can be related to the discussion on active and passive descriptions of rotations as elaborated by Shuster in the same review paper.